MIXED SPATIAL GAS FLOWS IN A LONGITUDINAL MAGNETIC FIELD

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Mixed spatial flows of infinite-conductivity gases in a longitudinal magnetic field are discussed. A small-perturbation method is used to find approximate equations for the magnetic potential of the perturbation velocity for all possible mixed flows; some of the particular integrals of these equations are obtained.

Mixed flows of ideally conducting gas in a longitudinal magnetic field were first studied for the planar case by Kogan [1], Peyret [2], and Seebass [3]. These studies showed that the imposition of the magnetic field leads to new mixed flows not previously encountered in ordinary gasdynamics. These results have recently been extended by Geffen [4], Tajiri [5], and Gorskii [6] to gas flows with axial symmetry.

1. We consider the steady-state three-dimensional adiabatic flow of a nonviscous, ideally conducting gas. We assume that the gas parameters are uniform and the magnetic field intensity H is parallel to the flow velocity **q** at an infinite distance upstream. Then, as Imai [7] has shown, H and **q** are collinear everywhere in the flow, and the gas motion is described by

$$\nabla \rho \mathbf{q} = 0, \quad \nabla \times (1 - A^{-2})\,\mathbf{q} = 0, \quad \frac{1}{2} q^2 + \frac{k}{k+1}\frac{p}{\rho} = \text{const}\,, \tag{1.1}$$

where ρ is the gas density, k is the adiabatic exponent, and A is the Alfvén number.

It follows from the second equation in system (1.1) that there exists a magnetic velocity potential Φ such that

$$\frac{\partial \Phi}{\partial x} = (1 - A^{-2})\, v_x, \quad \frac{\partial \Phi}{\partial y} = (1 - A^{-2})\, v_y, \quad \frac{\partial \Phi}{\partial z} = (1 - A^{-2})\, v_z\,, \tag{1.2}$$

where v_x, v_y, v_z are the velocity components parallel to the axes of the Cartesian coordinate system. Then the first equation (1.1) is converted to

$$\begin{gathered}\Phi_x{}^2\Phi_{xx} + \Phi_y{}^2\Phi_{yy} + \Phi_z{}^2\Phi_{zz} + 2\Phi_x\Phi_y\Phi_{xy} + 2\Phi_x\Phi_z\Phi_{xz} + 2\Phi_y\Phi_z\Phi_{yz} + \\ + L(q)\,(\Phi_x{}^2 + \Phi_y{}^2 + \Phi_z{}^2)(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}) = 0 \\ L(q) = M^{-2}(A^2 1 - M^2 - A^2)\,,\end{gathered} \tag{1.3}$$

where the subscripts on the potential denote differentiation with respect to the corresponding coordinate, and M is the Mach number. Exactly the same equation of motion, in different form, was obtained earlier by Imai [7] and Yur'ev [8].

Equation (1.3) is a differential equation of the mixed type; as is not difficult to see, its type changes at the surface at which M = 1, A = 1, $M^2 + A^2 = 1$. Here, as in the planar and axially symmetric cases, there are therefore three transition regions in which the flow is highly nonlinear, so the ordinary linearization method [9] cannot be applied in these regions. Nonlinearity is also displayed near the parabolic-degeneracy surface M = A = 1 of Eq. (1.3). We seek approximate equations of motion for each transition region.

2. The small-perturbation method yields simpler approximate equations of motion in the transition regions, which nevertheless retain the nonlinearity. We set

$$v_x = q_* (1 + u), \quad v_y = q_* v, \quad v_z = q_* w\,, \tag{2.1}$$

where u, v, and w are the components of the perturbation velocity, and the asterisk subscript denotes the critical value of the given quantity corresponding to some transition mode. Then derivatives (1.2) can be written

$$\Phi_x = q_* (1 - A^{-2})(1 + u), \quad \Phi_y = q_* (1 - A^{-2}) v, \quad \Phi_z = q_* (1 - A^{-2}) w\,. \tag{2.2}$$

We now introduce the magnetic potential φ of the perturbation velocity:

$$\Phi = q_* [(1 - A_*^{-2})x + \varphi], \quad \Phi_x = q_* (1 - A_*^{-2} + \varphi_x),$$
$$\Phi_y = q_* \varphi_y, \quad \Phi_z = q_* \varphi_z . \tag{2.3}$$

Comparing (2.2) and (2.3), we find

$$\varphi_x = u + A_*^{-2} [1 - (1 + u)\rho\rho_*^{-1}], \quad \varphi_y = (1 - A_*^{-2}\rho\rho_*^{-1})v$$
$$\varphi_z = (1 - A_*^{-2}\rho\rho_*^{-1})w . \tag{2.4}$$

To find the approximate equation for any transition mode, we substitute the values in (2.3) into Eq. (1.3); the function L(q) is also expressed, by means of Eqs. (2.4), in terms of the derivatives of the potential φ, and the leading terms are retained in it.

For the transonic region ($M \to 1$), assuming that

$$x = O(1), \quad y = O(\varepsilon^{-1/3}), \quad z = O(\varepsilon^{-1/3}), \quad \varphi = O(\varepsilon^{2/3}) , \tag{2.5}$$

we have

$$\varphi_x = u, \quad \varphi_y = (1 - A_*^{-2})v, \quad \varphi_z = (1 - A_*^{-2})w . \tag{2.6}$$

and the transonic equation of motion becomes

$$(k + 1)(1 - A_*^{-2})\varphi_x\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 ; \tag{2.7}$$

in Eqs. (2.5) ε is a small parameter. When the magnetic field disappears ($A_* = \infty$), this equation converts to the familiar Kármán equation.

In the hypercritical region ($M^2 + A^2 \to 1$), introducing the values.

$$x = O(1), \quad y = O(\varepsilon^{1/3}), \quad z = O(\varepsilon^{1/3}), \quad \varphi = O(\varepsilon^{4/3}) , \tag{2.8}$$

we find

$$\varphi_x = {}^1/_2 m [3 + (k - 2)M_*^2]u^2, \quad \varphi_y = - mv, \quad \varphi_z = - mw , \tag{2.9}$$
$$m = M_*^2 (1 - M_*^2)^{-1} ;$$

from (1.3) we find the hypercritical-flow equation:

$$n\varphi_x (\varphi_{yy} + \varphi_{zz})^2 - \varphi_{xx}^2 = 0, \quad n = 2M_*^{-2} (1 - M_*^2)^{-1} [3 + (k - 2)M_*^2] . \tag{2.10}$$

In the trans-Alfvén region ($A \to 1$) we set

$$x = O(1), \quad y = O(\varepsilon^{-1}), \quad z = O(\varepsilon^{-1}), \ \varphi = O(\varepsilon^2) ; \tag{2.11}$$

then we have

$$\varphi_x = M_*^2 W, \quad \varphi_y = M_*^2 vW, \quad \varphi_z = M_*^2 wW,$$
$$W = u + {}^1/_2 (v^2 + w^2) ,$$

and the approximate equation for the trans-Alfvén flow becomes

$$[(1 - M_*^{-2})\varphi_x^3 - \varphi_y^2 - \varphi_z^2]\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} +$$
$$+ 2\varphi_x\varphi_z\varphi_{xz} - \varphi_x^2 (\varphi_{yy} + \varphi_{zz}) = 0 . \tag{2.12}$$

Finally, in the transonic-(trans-Alfvén) region ($M \to 1$, $A \to 1$), we have

$$x = O(1), \quad y = O(\varepsilon^{-1}), \quad z = O(\varepsilon^{-1}),$$
$$\varphi = O(\varepsilon), \ \varphi_x = u, \quad \varphi_y = uv, \quad \varphi_z = uw ,$$

and the equation of motion becomes

$$(k+1)\varphi_x^4 - \varphi_y^2 - \varphi_z^2]\varphi_{xx} + 2\varphi_x\varphi_y\varphi_{xy} + 2\varphi_x\varphi_z\varphi_{xz} - \varphi_x^2(\varphi_{yy} + \varphi_{zz}) = 0. \tag{2.13}$$

It is easy to see that the corresponding equations for plane [3] and axially symmetric [6] mixed flows are obtained from Eqs. (2.7), (2.10), (2.12), and (2.13) as particular cases.

3. We discuss some particular solutions of these equations of motion in the transition regions. We first consider the transonic equations (2.7); it has the integral

$$\begin{aligned}\varphi = \alpha x^2 + \lambda\,[(\alpha^2 + B)\,y^2 + (\alpha^2 - B)\,z^2]\,x + \lambda^2\,\{[{}^1/_6\alpha\,(\alpha^2 + B) + D]y^4 + \\ + [{}^1/_6\alpha\,(\alpha^2 - B) + D]\,z^4 - 6Dy^2z^2 + C\,(y^2 - z^2)\},\end{aligned} \tag{3.1}$$

which is very similar to the solution of Ryzhov [10]. Here α, B, C, D are arbitrary constants, and $\lambda = (k+1)(1 - A_*^{-2})$. Since only even powers of y and z appear in Eq. (3.1), the nozzle flow which they describe has two symmetry planes, y = 0 and z = 0. Solution (3.1) contains as particular cases planar (for $B = \alpha^2$, C = D = 0) and axially symmetric (for B = C = D = 0) flows through a nozzle.

We find the projection u of the perturbation velocity on the x-axis:

$$u = \varphi_x = 2\alpha x + \lambda\,[(\alpha^2 + B)\,y^2 + (\alpha^2 - B)\,z^2]. \tag{3.2}$$

We see that the constant α is proportional to the velocity gradient along the nozzle axis (the x-axis). We set $\alpha > 0$; this means that the flow is accelerated in the direction of increasing x. From (3.2) we easily find the shape of the sonic surface u = 0:

$$x = -{}^1/_2\alpha^{-1}\,(k+1)(1 - A_*^{-2})\,[(\alpha^2 + B)\,y^2 + (\alpha^2 - B)\,z^2]. \tag{3.3}$$

For $|B| > \alpha^2$, this is accordingly an elliptic paraboloid; for $|B| = \alpha^2$, it is a parabolic cylinder; and for $|B| > \alpha^2$, it is a hyperbolic paraboloid. Now, using Eqs. (2.6), we find the surfaces v = 0 and w = 0 from (3.1):

$$y = 0,\ x = -\lambda\,(\alpha^2 + B)^{-1}\,\{[{}^1/_3\alpha\,(\alpha^2 + B) + 2D]\,y^2 - 6Dz^2 + C\}\ \ (v = 0) \tag{3.4}$$

$$z = 0,\ x = -\lambda\,(\alpha^2 - B)^{-1}\,\{[{}^1/_3\alpha\,(\alpha^2 - B) + 2D]\,z^2 - 6Dy^2 - C\}\ \ (w = 0). \tag{3.5}$$

These surfaces may accordingly be elliptic and hyperbolic paraboloids or parabolic cylinders and planes, which do not in general pass through the center of the nozzle (x = y = z = 0).

To find the characteristics x = x(y, z), we have

$$\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 = \lambda\varphi_x = 2\lambda\alpha x + \lambda^2\,[(\alpha^2 + B)\,y^2 + (\alpha^2 - B)z^2].$$

It follows that the characteristics passing through the center of the nozzle are four surfaces:

$$x = {}^1/_4(k+1)(1 - A_*^{-2})\,[(\alpha \pm \sqrt{5\alpha^2 + 4B})\,y^2 + (\alpha \pm \sqrt{5\alpha^2 - 4B})\,z^2]. \tag{3.6}$$

Interestingly, for $A_* > 1$ (super-Alfvén flow) the shape of the sonic surface and the characteristics are qualitatively the same as in ordinary transonic gasdynamics [10]. For $A_* < 1$ (sub-Alfvén flow), on the other hand, these surfaces are directed opposite those of the previous case.

We now write down a particular solution for the hypercritical-flow equation (2.10):

$$\varphi = {}^1/_{12}\alpha^2x^3 + n^{-1/2}\,[({}^1/_4\alpha + B)\,y^2 + ({}^1/_4\alpha - B)\,z^2], \tag{3.7}$$

where α and B are new arbitrary constants. This equation also describes the flow near a transition surface ($M^2 + A^2 = 1$ of a nozzle with two symmetry planes. In particular, for $B = \alpha/4$ and B = 0, we find from (3.7) integrals governing the flow through planar and axially symmetric nozzles, respectively [6]. Using Eqs. (2.9), we find the components u, v, and w:

$$u = \alpha n^{-1/2} M_*^{-2}x,\ v = 2n^{-1/2}\,(1 - M_*^{-2})\,({}^1/_4\alpha + B)\,y,\ w = 2n^{-1/2}\,(1 - M_*^{-2})\,({}^1/_4\alpha - B). \tag{3.8}$$

It follows that the transition surface (u = 0) is the x = 0 plane, while v = 0 and w = 0 at the y = 0 and z = 0 planes.

To find the characteristics F(x, y, z) = 0 we have

$$F_x^2 = \sqrt{n\varphi_x}\,(F_y^2 + F_z^2). \quad (3.9)$$

Since the derivative φ_x of (3.7) is equal to the φ_x found for the planar or axially symmetric cases, the characteristics coming from the transition surface are also the same. These are accordingly surfaces of rotation formed by semicubic parabolas having cuspidal points on the $x=0$ plane and symmetry axes parallel to the x-axis:

$$x^3 - {}^9/_2\alpha^{-1}n^{-1/2}\,[(y + c_1)^2 + (z + c_2)^2] = 0, \quad (3.10)$$

where c_1, c_2 are arbitrary constants.

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